ME 202 : Advanced Mechanics of Solids

Transformation of Stress in Three Dimensions

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1 Introduction

The most general state of stress at a given point may be represented by six components. Three of these components are normal stresses, represented by σ_x, σ_y and σ_z and the other three represented by τ_{xy}, τ_{yz} and τ_{xz} are shear stresses. The same state of stress can be represented by a different set of components if the coordinate axes are rotated as shown in Figure 1. Stress transformation deals with determination of the different components of stress under a rotation of coordinate axes.

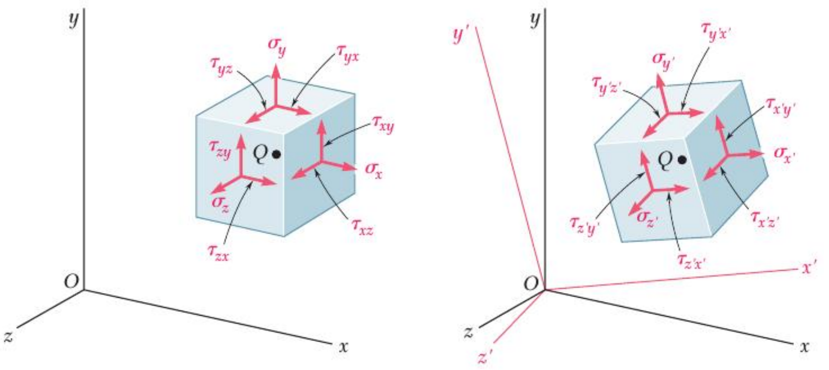

Figure 1: General state of stress at a point.

2 Transformation of Stresses

Let (x,y,z) and (X,Y,Z) denote two rectangular coordinate systems with a common origin as shown in Figure 2. Let the direction cosines of X-axis with respect to coordinate system (x,y,z) be l_x, l_y and l_z respectively. Similarly m_x, m_y and m_z are the direction cosines of Y-axis and n_x, n_y and n_z are the direction cosines of Z-axis. These are listed in Table 1 and it should be noted that, here θ_{xX} denotes angle between x-axis and X-axis.

Table 1: Direction Cosines

	x	y	z
X	$l_x = \cos\theta_{xX}$	$l_y = \cos\theta_{yX}$	$l_z = \cos\theta_{zX}$
Y	$m_x = \cos\theta_{xY}$	$m_y = \cos\theta_{yY}$	$m_z = \cos\theta_{zY}$
Z	$n_x = \cos\theta_{xZ}$	$n_y = \cos\theta_{yZ}$	$n_z = \cos\theta_{zZ}$

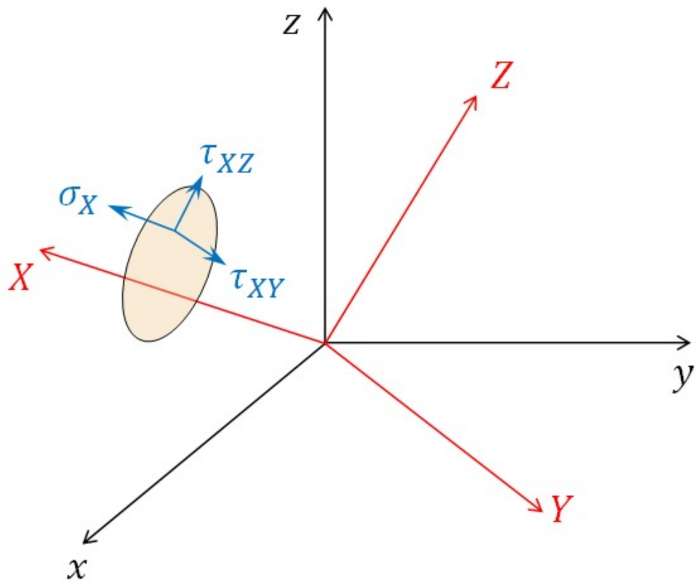

Figure 2: Stress components on plane perpendicular to transformed X-axis.

According to Cauchy stress equation, the stress vector on a plane with unit normal **n** with direction cosines n_x, n_y and $n_z$is given by

$$\overset{n}{\mathbf{T}} = \overset{n}{T}_x\mathbf{i} + \overset{n}{T}_y\mathbf{j} + \overset{n}{T}_z\mathbf{k}$$

where,

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Hence, the stress vector on a plane perpendicular to X-axis (Conveniently referred as X-plane), having direction cosines l_x, l_y and l_z is given by,

$$\overset{X}{\mathbf{T}} = \overset{X}{T}_x\mathbf{i} + \overset{X}{T}_y\mathbf{j} + \overset{X}{T}_z\mathbf{k}$$

where,

$$\overset{X}{T}_x = \sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z$$

$$\overset{X}{T}_y = \tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z$$

$$\overset{X}{T}_z = \tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z$$

The normal stress along X-direction can be obtained by taking the component of $\overset{X}{\mathbf{T}}$ along the direction of X-axis. i.e.

$$\sigma_{XX} = \overset{X}{\mathbf{T}} \bullet (l_x \mathbf{i} + l_y \mathbf{j} + l_z \mathbf{k})$$

$$\sigma_{XX} = \overset{X}{T}_x l_x + \overset{X}{T}_y l_y + \overset{X}{T}_z l_z$$

$$\sigma_{XX} = (\sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z) l_x + (\tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z) l_y + (\tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z) l_z$$

After rearranging the above expression, we obtain

$$\sigma_{XX} = {l_x}^2 \sigma_x + {l_y}^2 \sigma_y + {l_z}^2 \sigma_z + 2 l_x l_y \tau_{xy} + 2 l_y l_z \tau_{yz} + 2 l_z l_x \tau_{zx}$$

Similarly, using stress vectors of Y-plane and Z-plane,σ_{YY} and σ_{ZZ} can be obtained.

$$\sigma_{YY} = {m_x}^2 \sigma_x + {m_y}^2 \sigma_y + {m_z}^2 \sigma_z + 2 m_x m_y \tau_{xy} + 2 m_y m_z \tau_{yz} + 2 m_z m_x \tau_{zx}$$

$$\sigma_{ZZ} = {n_x}^2 \sigma_x + {n_y}^2 \sigma_y + {n_z}^2 \sigma_z + 2 n_x n_y \tau_{xy} + 2 n_y n_z \tau_{yz} + 2 n_z n_x \tau_{zx}$$

The shear stress component σ_{XY} is the component of the stress vector in the Y direction on a plane perpendicular to the X axis; that is, it is the Y component of the stress vector $\overset{X}{\mathbf{T}}$ acting on the plane perpendicular to the X axis.Thus, σ_{XY} may be evaluated by forming the scalar product of the vector $\overset{X}{\mathbf{T}}$ with a unit vector parallel to the Y axis, that is,

$$\tau_{XY} = \overset{X}{\mathbf{T}} \bullet (m_x \mathbf{i} + m_y \mathbf{j} + m_z \mathbf{k})$$

$$\tau_{XY} = \overset{X}{T}_x m_x + \overset{X}{T}_y m_y + \overset{X}{T}_z m_z$$

$$\tau_{XY} = (\sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z) m_x + (\tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z) m_y + (\tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z) m_z$$

$$\tau_{XY} = l_x m_x \sigma_x + l_y m_y \sigma_y + l_z m_z \sigma_z + (l_x m_y + l_y m_x) \tau_{xy} + (l_y m_z + l_z m_y) \tau_{yz} + (l_z m_x + l_x m_z) \tau_{zx}$$

Similar procedures also determine σ_{XY} and σ_{ZX} as,

$$\tau_{YZ} = m_x n_x \sigma_x + m_y n_y \sigma_y + m_z n_z \sigma_z + (m_x n_y + m_y n_x) \tau_{xy} + (m_y n_z + m_z n_y) \tau_{yz} + (m_z n_x + m_x n_z) \tau_{zx}$$

$$\tau_{ZX} = n_x l_x \sigma_x + n_y l_y \sigma_y + n_z l_z \sigma_z + (n_x l_y + n_y l_x) \tau_{xy} + (n_y l_z + n_z l_y) \tau_{yz} + (n_z l_x + n_x l_z) \tau_{zx}$$

3 Transformation Equation in Matrix Form

Written in matrix form, the transformation equation is :

$$[\sigma]_{XYZ} = [T][\sigma]_{xyz}[T]^T$$

where, $[T]$ is the transformation matrix and $[T]^T$ is the transpose of the transformation matrix given by:

$$[T] = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}$$

and

$$[T]^T = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

4 Example 1

Given the components of stress as $\sigma_x = 1, \sigma_y = 2, \sigma_z = -1, \tau_{xy} = 2, \tau_{yz} = 1$ and $\tau_{xz} = -3$. Transform these components based on a new x and y co-ordinates turned through 45^O in the anti clockwise direction and the z-axis unchanged.

Solution

Let the given state of stress is in the xyz co-ordinates and the new co-ordinates be XYZ. The given state of stress can be written in the tensor form as

$$[\sigma]_{xyz} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

The transformation matrix is given by,

$$[T] = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} = \begin{bmatrix} \cos 45 & \cos 45 & \cos 90 \\ \cos 135 & \cos 45 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying these values in equations derived in the previous section, we get
$\sigma_X = 3.499, \tau_{XY} = 0.499, \tau_{XZ} = -1.414$
$\sigma_Y = -0.499, \sigma_Z = -1, \tau_{YZ} = 2.828$
Alternatively;

$$[\sigma]_{XYZ} = [T][\sigma]_{xyz}[T]^T$$

$$[\sigma]_{XYZ} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.499 & 0.499 & -1.414 \\ 0.499 & -0.499 & 2.828 \\ -1.414 & 2.828 & -1.000 \end{bmatrix}$$