Analysis of Stress Part 1

Advanced Mechanics of Solids ME202

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Outline

Stress Analysis - Introduction

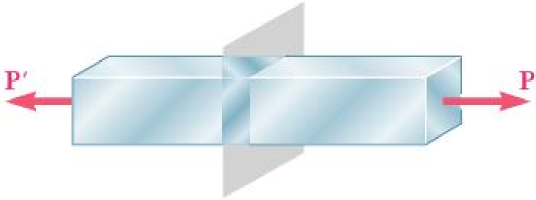
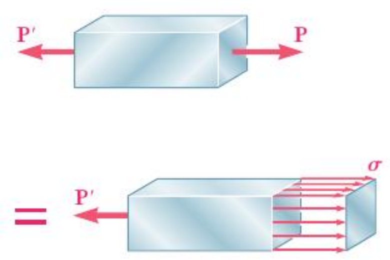


Figure: Rectangular rod subjected to axial-loads

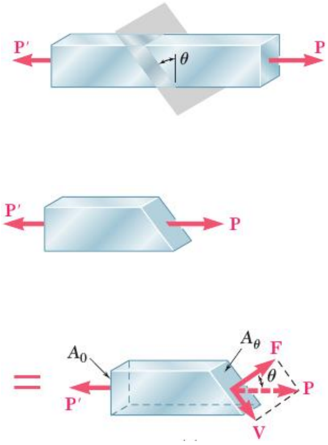
Stress Analysis - Introduction



- Let A_O, be the area of the rod
- Normal stress in the rod,

$$\sigma = \frac{P}{A_O}$$

Stress Analysis - Introduction



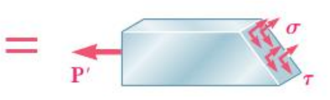
- Resolving P into components F and V, respectively normal and tangential to the section, we have

$$F = P\cos\theta, V = P\sin\theta$$

- Normal and shearing stresses are obtained by

$$\sigma = \frac{P\cos\theta}{A_\theta}, \tau = \frac{P\sin\theta}{A_\theta}$$

Stress Analysis - Introduction



- Here,

$$A_O = A_\theta \cos\theta$$

- Hence, Normal and shearing stresses are obtained by

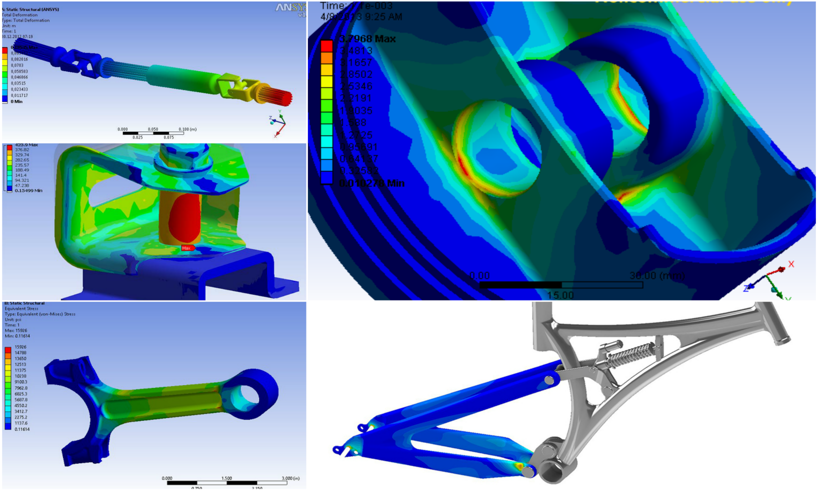
$$\sigma = \frac{P\cos^2\theta}{A_O}, \tau = \frac{P\sin\theta\cos\theta}{A_O}$$

Body Forces, Surface Forces

In general an arbitrary object (body) will be subjected to two types of forces-

- Body forces - Which act on each volume element of the body. Eg: Gravitational force, Electrostatic force, Magnetic force, Inertia force etc.
- Surface forces - Which act on the surface or area elements of the body. Forces acting on the actual boundaries are called surface tractions. Eg: Pressure force, Friction, Support reactions, Stress (Traction) etc.

Body Forces, Surface Forces and Stress Vector



Body Forces, Surface Forces and Stress Vector

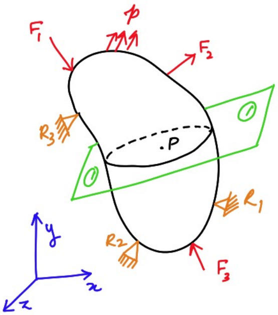


Figure: Body subjected to forces

Body Forces, Surface Forces and Stress Vector

- Let P be a point inside the body with coordinates (x,y,z)
- Let the body be cut in to two parts C and D by a plane passing through point P
- Then, each part C and D is in equilibrium under the action of the externally applied forces and internally distributed forces across the interface.

Body Forces, Surface Forces and Stress Vector

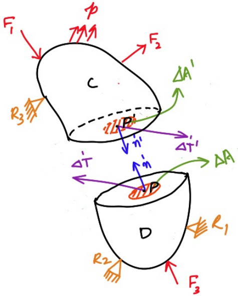


Figure: Free-body diagram of a body cut into two parts

Body Forces, Surface Forces and Stress Vector

- In part D, let ΔA be a small area surrounding the point P. In part C, the corresponding area at P' is $\Delta A'$
- The areas ΔA and $\Delta A'$ are distinguished by their outward normals $\overset{1}{\mathbf{n}}$ and $\overset{1}{\mathbf{n}}$'
- The action of part C on ΔA at point P can be represented by force vector $\Delta \overset{1}{\mathbf{T}}$ and the action of part D on $\Delta A'$ at P' can be represented by the force vector $\Delta \overset{1}{\mathbf{T}}$'
- As ΔA tends to zero, the ratio $\dfrac{\Delta \overset{1}{\mathbf{T}}}{\Delta A}$ tends to a definite limit, and the moment of forces acting on area vanishes in the limit.

Body Forces, Surface Forces and Stress Vector

- The limiting vector is written as

$$\lim_{\Delta A \to 0} \frac{\Delta \overset{1}{\mathbf{T}}}{\Delta A} = \frac{d\overset{1}{\mathbf{T}}}{dA} = \overset{1}{\mathbf{T}}$$

- Similarly, at point P'

$$\lim_{\Delta A \to 0} \frac{\Delta \overset{1}{\mathbf{T}}'}{\Delta A'} = \frac{d\overset{1}{\mathbf{T}}'}{dA'} = \overset{1}{\mathbf{T}}'$$

- Vectors $\overset{1}{\mathbf{T}}$ and $\overset{1}{\mathbf{T}}'$ are called **Stress Vectors**

Body Forces, Surface Forces and Stress Vector

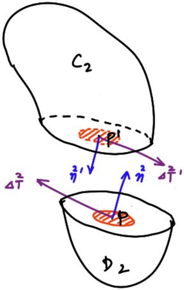


Figure: Body cut by another plane

Body Forces, Surface Forces and Stress Vector

- If the body is cut by a different plane 2-2 with outward drawn normals $\overset{2}{\mathbf{n}}$ and $\overset{2}{\mathbf{n}}'$ passing through same point P', then stress vector representing action of C2 on D2 will be represented by $\overset{2}{\mathbf{T}}$, i.e.
$$\overset{2}{\mathbf{T}} = \frac{\Delta \overset{2}{\mathbf{T}}'}{\Delta A'}$$
- In general, stress vector $\overset{1}{\mathbf{T}}$ acting at point P on a plane with outward drawn normal $\overset{1}{\mathbf{n}}$ will be different from stress vector $\overset{2}{\mathbf{T}}$ acting at the same point P, but on a plane with outward normal $\overset{2}{\mathbf{n}}$

State of Stress at a Point

- Since an infinite number of can be drawn through a point, we can get an infinite number of stress vectors at a given point.
- Each stress vector characterised by the corresponding plane on which it is acting
- The totality of all stress vectors acting on every possible plane passing through the point is defined to be **state of stress at the point**.

`However, if the stress vectors acting on three mutually perpendicular planes passing through the point are known, the stress vector acting on any other arbitrary plane at that point can be determined.`

Normal and Shear stress components

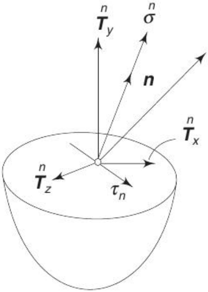


Figure: Resultant Stress Vector - Normal and Shear stress components

- The stress vector $\overset{n}{\mathbf{T}}$ at point P can be resolved in to two components.
- The component along the normal **n** is called normal stress and is denoted by $\overset{n}{\sigma}$
- The component perpendicular to **n** is denoted by $\overset{n}{\tau}$
- $$|\overset{n}{\mathbf{T}}|^2 = \overset{n}{\sigma}{}^2 + \overset{n}{\tau}{}^2$$

Normal and Shear stress components

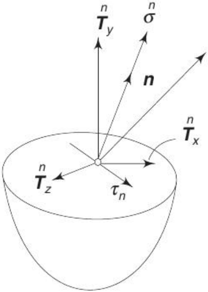


Figure: Resultant Stress Vector - x,y and z components

- Stress vector $\overset{n}{\mathbf{T}}$ can also be resolved in to three components parallel to $x, y \& z$ axes.
- If these components are $\overset{n}{\mathbf{T}}_x, \overset{n}{\mathbf{T}}_y \& \overset{n}{\mathbf{T}}_z$

$$|\overset{n}{\mathbf{T}}|^2 = \overset{n}{\mathbf{T}}_x^2 + \overset{n}{\mathbf{T}}_y^2 + \overset{n}{\mathbf{T}}_z^2$$

Rectangular Stress Components

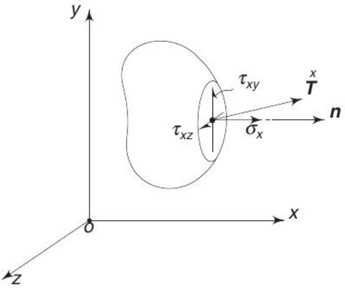


Figure: Stress components on x plane

- Let the body be cut by a plane perpendicular to x-axis. The resultant stress vector at P acting on this will be $\overset{x}{\mathbf{T}}$.
- The component parallel to the x axis, being normal to the plane, will be denoted by σ_x (instead of by $\overset{x}{\sigma}$).
- The components parallel to the y and z axes are shear stress components and are

Rectangular Stress Components

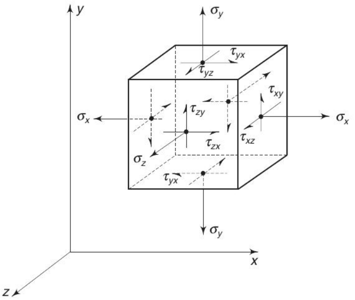


Figure: Rectangular stress components

At any point P, one can draw three mutually perpendicular planes, the x plane, the y plane and the z plane.

- $\sigma_x, \tau_{xy}, \tau_{xz}$ on x plane
- $\sigma_y, \tau_{yx}, \tau_{yz}$ on y plane
- $\sigma_z, \tau_{zx}, \tau_{zy}$ on z plane

These components are shown acting on a small rectangular element surrounding the point P.

Equality of Cross Shears

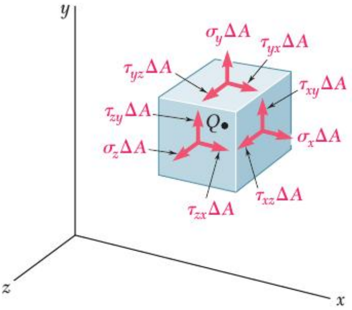


Figure: Components of Stress

- Out of the nine rectangular stress components

$$\sigma_x, \tau_{xy}, \tau_{xz},$$
$$\sigma_y, \tau_{yx}, \tau_{yz},$$
$$\sigma_z, \tau_{zx}, \tau_{zy}$$

, only six are independent.

- i.e. $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}$ $, \tau_{xz} = \tau_{zx}$

Equality of Cross Shears

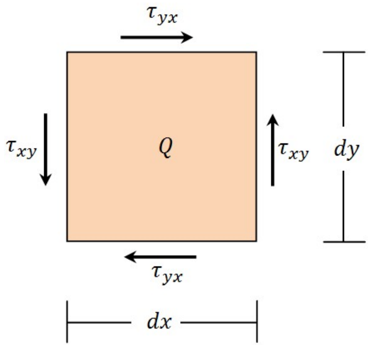


Figure: Front View of Element with Components of Shear stress alone

- Taking moments about point Q, for equilibrium, $\Sigma M_Q = 0$
-
$$2\tau_{xy} dydz \frac{dx}{2} - 2\tau_{yx} dxdz \frac{dy}{2} = 0$$
$$i.e. \tau_{xy} = \tau_{yx}$$
- Similarly, it can be proved that $\tau_{yz} = \tau_{zy}, \tau_{xz} = \tau_{zx}$
- $\tau_{xy} and \tau_{yx}, \tau_{xz} and \tau_{zx}$ etc. are called **Complimentary Shear Stresses**

Equality of Cross Shears

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Equality of Cross Shears

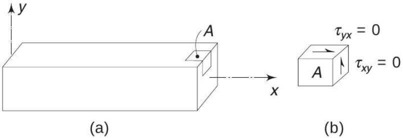


Figure: (a) Element with free surface; (b) Cross shears being zero

- Equality of cross shears can be used to prove that a shear cannot cross a free boundary.
- For a beam of rectangular cross-section as shown in Figure $\tau_{xy} = 0$ if the top surface is free of stresses

Stress Components in Cylindrical/Polar Co-ordinates

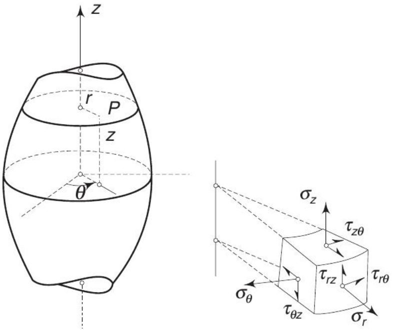


Figure: (a) Cylindrical coordinates of a point (b) Stresses on an element

Stress components in Cylindrical Co-ordinates

$$\sigma = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix}$$

Stress Components on an Arbitrary Plane

The stress vector acting on any other arbitrary plane at a point can be determined, If The stress vectors acting on three mutually perpendicular planes passing through the point are known by using **Cauchy's stress formulae**

Stress Components on an Arbitrary Plane

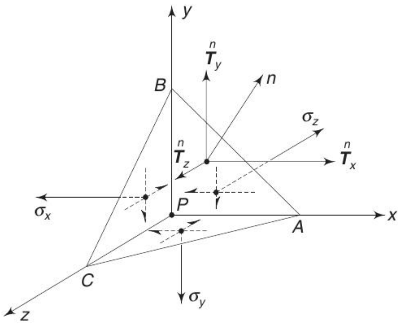


Figure: Tetrahedron at point P

- Consider the Tetrahedron as shown.
- Let **n** be the normal to inclined face.
- Let h be the perpendicular distance from P to inclined face.
- Let the body force components in x, y and z directions be γ_x, γ_y and γ_z respectively, per unit volume.

Stress Components on an Arbitrary Plane

For equilibrium of the tetrahedron, the sum of the forces in x, y and z directions must individually vanish. Thus, for equilibrium in x direction

$$\overset{n}{T}_x A - \sigma_x A n_x - \tau_{yx} A n_y - \tau_{zx} A n_z + \frac{1}{3} A h \gamma_x = 0$$

Cancelling A,

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z - \frac{1}{3} h \gamma_x$$

Similarly for y and z directions,

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z - \frac{1}{3} h \gamma_y$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z - \frac{1}{3} h \gamma_z$$

Stress Components on an Arbitrary Plane

In the limit as h tends to zero,

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$$

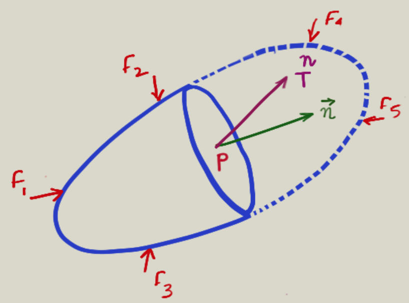
$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

This Equation is known as **Cauchy's stress formula.**

Cauchy's Stress Formula

Cauchy's Stress Formula can be expressed in matrix form as,

$$\begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \\ \overset{n}{T}_z \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$



As a Tensor formula,

$$\overset{n}{\mathbf{T}} = \boldsymbol{\sigma}\mathbf{n}$$

- If $\overset{n}{\mathbf{T}}$ is the resultant stress vector on plane ABC, we have,

$$\overset{n}{\mathbf{T}} = \overset{n}{T}_x \mathbf{i} + \overset{n}{T}_y \mathbf{j} + \overset{n}{T}_z \mathbf{k}$$

$$|\overset{n}{\mathbf{T}}|^2 = \overset{n}{T}_x^{\,2} + \overset{n}{T}_y^{\,2} + \overset{n}{T}_z^{\,2}$$

- If σ_n and τ_n are the normal and shear stress components, we have

$$|\overset{n}{\mathbf{T}}|^2 = {\sigma_n}^2 + {\tau_n}^2$$

- Normal stress, σ_n

$$\sigma_n = n_x \overset{n}{T}_x + n_y \overset{n}{T}_y + n_z \overset{n}{T}_z$$

$$\sigma_n = {n_x}^2 \sigma_x + {n_y}^2 \sigma_y + {n_z}^2 \sigma_z + 2 n_x n_y \tau_{xy} + 2 n_y n_z \tau_{yz} + 2 n_z n_x \tau_{zx}$$

Example-1

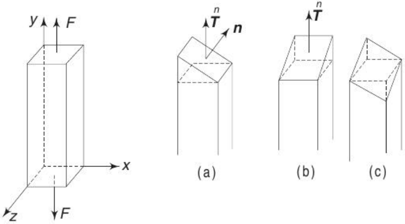


Figure: Example-1

A rectangular steel bar having a cross-section 2 cm x 3 cm is subjected to a tensile force of 6000 N. If the axes are chosen as shown in Figure below, determine the normal and shear stresses on a plane whose normal has the following direction cosines:

1. $n_x = n_y = \dfrac{1}{\sqrt{2}}, n_z = 0$
2. $n_x = 0, n_y = n_z = \dfrac{1}{\sqrt{2}}$
3. $n_x = n_y = n_z = \dfrac{1}{\sqrt{3}}$

Example-1 - Solution

Area of section, $A = 2cmX3cm = 6cm^2$
Average stress on a section perpendicular to y-axis,
i.e. $\sigma_y = \frac{6000}{6} = 1000N/cm^2$
All other stress components are zero on y-plane.
$\sigma_x = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$
1) Using Cauchy's sress equations,

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z = 0$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z = \frac{1000}{\sqrt{2}} N/cm^2$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = 0$$

$$\sigma_n = n_x \overset{n}{T}_x + n_y \overset{n}{T}_y + n_z \overset{n}{T}_z$$

$$\sigma_n = \frac{1}{\sqrt{2}} \times 0 + \frac{1}{\sqrt{2}} \times \frac{1000}{\sqrt{2}} + 0 \times 0 = \frac{1000}{2} = 500N/cm^2$$

$$\tau_n{}^2 = |\overset{n}{\mathbf{T}}|^2 - \sigma_n{}^2$$

$$|\overset{n}{\mathbf{T}}|^2 = \overset{n}{T}_x{}^2 + \overset{n}{T}_y{}^2 + \overset{n}{T}_z{}^2 = 0 + (\frac{1000}{\sqrt{2}})^2 + 0 = 500000$$

$$\tau_n{}^2 = 500000 - 500^2 = 250000$$

$$\tau_n = 500N/cm^2$$

2)

$$\overset{n}{\mathbf{T}}_x = 0, \overset{n}{\mathbf{T}}_z = \frac{1000}{\sqrt{2}}, \overset{n}{\mathbf{T}}_y = 0$$

$$\sigma_n = 500N/cm^2, and \tau_n = 500N/cm^2$$

3)

$$\overset{n}{\mathbf{T}}_x = 0, \overset{n}{\mathbf{T}}_y = \frac{1000}{\sqrt{3}}, \overset{n}{\mathbf{T}}_y = 0$$

$$\sigma_n = \frac{1000}{3}N/cm^2, and \tau_n = 417N/cm^2$$