Analysis of Strain

Advanced Mechanics of Solids ME202

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Outline

Displacement Field

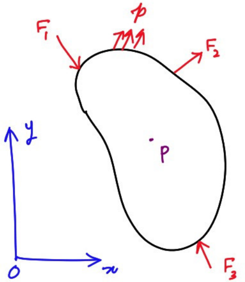

Figure: Solid body under external forces

- Displacement Field is used for mathematical description of shape change in solids.
- Figure represents a solid body under the action of external forces.
- Every point within the body moves as the load is applied.

Displacement Field

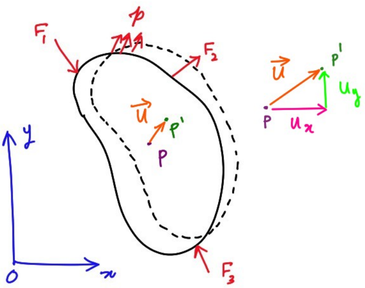

Figure: Solid body after deformation

- Due to deformation the point P is displaced to P',the vector segment **PP'** is called **the displacement vector** and is denoted by **u**
- **For 2D**
  $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$
  Where $u_x = u_x(x, y)$ and
  $u_y = u_y(x, y)$
- **Similarly for 3D**
  $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$
  $u_x = u_x(x, y, z), u_y = u_y(x, y, z)$
  and $u_z = u_z(x, y, z)$

Displacement Field: Example

The displacement field for a body is given by $\mathbf{u} = (x^2 + y)\mathbf{i} + (3 + z)\mathbf{j} + (x^2 + 2y)\mathbf{k}$. What is the deformed position of a point originally at (3, 1, –2)?

Displacement Field: Example

The displacement field for a body is given by $\mathbf{u} = (x^2 + y)\mathbf{i} + (3 + z)\mathbf{j} + (x^2 + 2y)\mathbf{k}$. What is the deformed position of a point originally at (3, 1, –2)?

Solution:

Displacement vector **u** at (3,1,-2)is,

$\mathbf{u} = (3^2 + 1)\mathbf{i} + (3 - 2)\mathbf{j} + (3^2 + 2)\mathbf{k}$

$i.e.\mathbf{u} = 10\mathbf{i} + \mathbf{j} + 11\mathbf{k}$

The initial position vector of the point, $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Position vector of the point after deformation,

$\mathbf{r'} = \mathbf{r} + \mathbf{u} = 13\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$

Strains at A Point

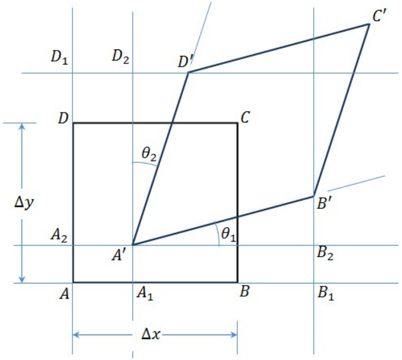

Figure: Deformation of a rectangular element

$$AA_1 = A_2A' = u_x$$
$$BB_1 = u_x + \frac{\partial u_x}{\partial x}\Delta x$$
$$AA_2 = A_1A' = u_y$$
$$DD_1 = u_y + \frac{\partial u_y}{\partial y}\Delta y$$
$$B_1B' = u_y + \frac{\partial u_y}{\partial x}\Delta x$$
$$D_1D' = u_x + \frac{\partial u_x}{\partial y}\Delta y$$

Normal Strain ϵ_x

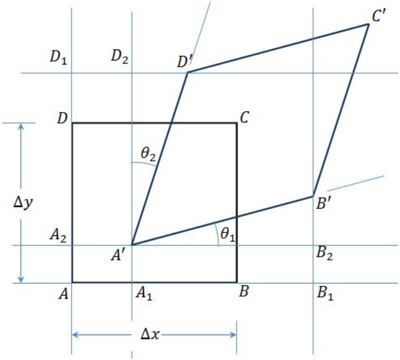

Figure: Deformation of a rectangular element

$$\epsilon_{xx} \quad \text{or} \quad \epsilon_x = \lim_{\Delta x \to 0} \frac{A'B' - AB}{AB}$$

$$\approx \lim_{\Delta x \to 0} \frac{A'B_2 - AB}{AB}$$

$$= \lim_{\Delta x \to 0} \frac{AB + BB_1 - AA_1 - AB}{AB}$$

$$= \lim_{\Delta x \to 0} \frac{BB_1 - AA_1}{AB}$$

$$= \lim_{\Delta x \to 0} \frac{u_x + \frac{\partial u_x}{\partial x}\Delta x - u_x}{\Delta x} = \frac{\partial u_x}{\partial x}$$

$$\epsilon_x = \frac{\partial u_x}{\partial x}$$

Normal Strain ϵ_y

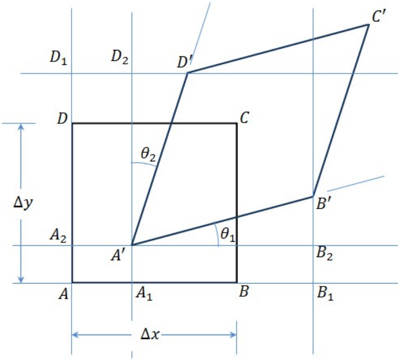

Figure: Deformation of a rectangular element

$$\epsilon_{yy} \quad \text{or} \quad \epsilon_y = \lim_{\Delta x \to 0} \frac{A'D' - AD}{AD}$$

$$\approx \lim_{\Delta y \to 0} \frac{A'D_2 - AD}{AD}$$

$$= \lim_{\Delta y \to 0} \frac{AD + DD_1 - AA_2 - AD}{AD}$$

$$= \lim_{\Delta y \to 0} \frac{DD_1 - AA_2}{AD}$$

$$= \lim_{\Delta y \to 0} \frac{u_y + \dfrac{\partial u_y}{\partial y}\Delta y - u_y}{\Delta y} = \frac{\partial u_y}{\partial y}$$

$$\epsilon_y = \frac{\partial u_y}{\partial y}$$

Engineering Shear Strain γ_{xy}

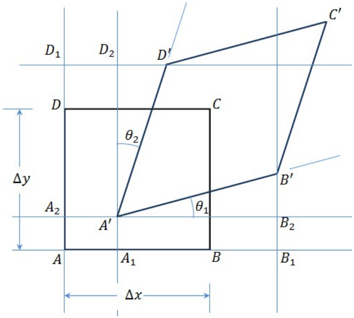

Figure: Deformation of a rectangular element

$$\gamma_{xy} \quad \text{or} \quad \gamma_{yx} = \lim_{\Delta x, \Delta y \to 0} \theta_1 + \theta_2$$

$$\approx \lim_{\Delta x, \Delta y \to 0} \left(\frac{B_2 B'}{A' B_2} + \frac{D_2 D'}{A' D_2} \right)$$

$$= \lim_{\Delta x, \Delta y \to 0} \left(\frac{\frac{\partial u_y}{\partial x} \Delta x}{\Delta x + \frac{\partial u_x}{\partial x} \Delta x} + \frac{\frac{\partial u_x}{\partial y} \Delta y}{\Delta y + \frac{\partial u_y}{\partial y} \Delta y} \right)$$

$$= \lim_{\Delta x, \Delta y \to 0} \left(\frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}} + \frac{\frac{\partial u_x}{\partial y}}{1 + \frac{\partial u_y}{\partial y}} \right)$$

$$\approx \lim_{\Delta x, \Delta y \to 0} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right)$$

$$\gamma_{xy} = \gamma_{yx} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right)$$

Rectangular Strain Components in 3-Dimensions

Linear or Normal Strains

$$\epsilon_x = \frac{\partial u_x}{\partial x}$$

$$\epsilon_y = \frac{\partial u_y}{\partial y}$$

$$\epsilon_z = \frac{\partial u_z}{\partial z}$$

Engineering Shear Strains

$$\gamma_{xy} = \gamma_{yx} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\gamma_{yz} = \gamma_{zy} = \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

$$\gamma_{zx} = \gamma_{xz} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

Tensorial Strain Components

Strain Tensor

$$\begin{bmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{bmatrix}$$

Where,

ϵ_{ij} are called Cauchy's Shear Strain or Tensorial Shear Strain

$$\epsilon_{xy} \quad \text{or} \quad \epsilon_{yx} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\gamma_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\epsilon_{yz} \quad \text{or} \quad \epsilon_{zy} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\gamma_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

$$\epsilon_{zx} \quad \text{or} \quad \epsilon_{xz} = \frac{1}{2}\gamma_{zx} = \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

Strain Tensor

Strain Tensor

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{bmatrix} = \begin{bmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_z \end{bmatrix} \quad \text{is a Tensor}$$

$$\text{Engineering Strain} \quad \begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix} \quad \text{is not a Tensor}$$

Strain-Displacement Relationship - Excercises

Consider the displacement field $[y^2\mathbf{i} + 3yz\mathbf{j} + (4 + 6x^2)\mathbf{k}]10^{-2}$
What are the rectangular strain components at point P (1,0,2)?

Strain-Displacement Relationship - Excercises

Consider the displacement field $[y^2\mathbf{i} + 3yz\mathbf{j} + (4 + 6x^2)\mathbf{k}]10^{-2}$
What are the rectangular strain components at point P (1,0,2)?
Solution:
$u_x = y^2$, $u_y = 3yz$ and $u_z = (4 + 6x^2)$
Linear Strains:

$$\epsilon_x = \frac{\partial u_x}{\partial x} = \frac{\partial (y^2)}{\partial x} 10^{-2} = 0$$

$$\epsilon_y = \frac{\partial u_y}{\partial y} = \frac{\partial (3yz)}{\partial y} 10^{-2} = 3z10^{-2} \quad \text{at P,} \quad \epsilon_y = 6 \times 10^{-2}$$

$$\epsilon_z = \frac{\partial u_z}{\partial z} = \frac{\partial (4 + 6x^2)}{\partial y} 10^{-2} = 0$$

Strain-Displacement Relationship - Excercises

Shear Strains:

$$\gamma_{xy} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) = (0 + 2y) \times 10^{-2}$$

$$\gamma_{xy} \quad \text{at P,} \quad = 0$$

$$\gamma_{yz} = \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) = (0 + 3y) \times 10^{-2}$$

$$\gamma_{yz} \quad \text{at P,} \quad = 0$$

$$\gamma_{zx} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = (0 + 12x) \times 10^{-2}$$

$$\gamma_{zx} \quad \text{at P,} \quad = 12 \times 10^{-2}$$

Analogy between Stress and Strain Tensors

- Diagonal elements of stress tensor represent normal stress. Diagonal elements of strain tensor represent normal strain.
- Off-diagonal elements of the stress tensor represent the shear stresses on planes parallel to coordinate planes. The off-diagonal elements of the strain tensor are half of the shear strains on planes parallel to coordinate planes.
- Both stress and strain tensors are symmetric
- The extreme values of normal stresses called the principal stresses are the Eigen values of the stress tensor. The extreme values of normal strains from a point are given by the Eigen values of the strain tensor.

- The planes of principal stresses (principal planes)are given by Eigen vectors of stress tensor. Directions of principal strains(principal directions) are the Eigen vectors of strain tensor.
- The transformation rule is equally applicable for both stress and strain tensors.
- Stress and strain tensors are related through a fourth order tensor consisting of properties of material of the body

Linear Strain a Point along a Given Direction

If the state of strain at a point P is defined by the strain tensor,

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{bmatrix}$$

The linear strain at the point P in the direction PQ with direction cosines n_x, n_y, and n_z, is given by,

$$\epsilon_{PQ} = {n_x}^2\epsilon_x + {n_y}^2\epsilon_y + {n_z}^2\epsilon_z + 2n_x n_y \epsilon_{xy} + 2n_y n_z \epsilon_{yz} + 2n_z n_x \epsilon_{zx}$$

Strain along a perticular Direction - Excercise

The following state of strain exists at a point P,

$$[\epsilon_{ij}] = \begin{bmatrix} 0.02 & -0.04 & 0 \\ -0.04 & 0.06 & -0.02 \\ 0 & -0.02 & 0 \end{bmatrix}$$

In the direction PQ having direction cosines $n_x = 0.6$, $n_y = 0$ and $n_z = 0.8$, determine ϵ_{PQ}

Strain along a perticular Direction - Excercise

The following state of strain exists at a point P,

$$[\epsilon_{ij}] = \begin{bmatrix} 0.02 & -0.04 & 0 \\ -0.04 & 0.06 & -0.02 \\ 0 & -0.02 & 0 \end{bmatrix}$$

In the direction PQ having direction cosines $n_x = 0.6$, $n_y = 0$ and $n_z = 0.8$, determine ϵ_{PQ}

Solution:

$$\epsilon_{PQ} = {n_x}^2\epsilon_x + {n_y}^2\epsilon_y + {n_z}^2\epsilon_z + 2n_x n_y \epsilon_{xy} + 2n_y n_z \epsilon_{yz} + 2n_z n_x \epsilon_{zx}$$

$$\epsilon_{PQ} = 0.02(0.36)+0.06(0)+0(0.64)-0.04(0)-0.02(0)+0(0.96) = 0.007$$

Principal Strains and Directions

A in the case of Principal Stresses, in order to obtain Principal strains and directions, a system of linear homogeneous equations can be formed.

$$(\epsilon_x - \epsilon)n_x + \epsilon_{yx}n_y + \epsilon_{zx}n_z = 0$$

$$\epsilon_{xy}n_x + (\epsilon_y - \epsilon)n_y + \epsilon_{zy}n_z = 0$$

$$\epsilon_{xz}n_x + \epsilon_{yz}n_y + (\epsilon_z - \epsilon)n_z = 0$$

Principal Strains are given by,

$$\begin{vmatrix} (\epsilon_x - \epsilon) & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & (\epsilon_y - \epsilon) & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & (\epsilon_z - \epsilon) \end{vmatrix} = 0$$

Principal Strains and Directions

On expanding the above determinant, we get,

$$\epsilon^3 - J_1\epsilon^2 + J_2\epsilon - J_3 = 0$$

Where, J_1, J_2 and J_3 are **Strain Invariants**

$$J_1 = \epsilon_x + \epsilon_y + \epsilon_z$$

$$J_2 = \begin{vmatrix} \epsilon_x & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_y \end{vmatrix} + \begin{vmatrix} \epsilon_y & \epsilon_{zy} \\ \epsilon_{yz} & \epsilon_z \end{vmatrix} + \begin{vmatrix} \epsilon_x & \epsilon_{xz} \\ \epsilon_{zx} & \epsilon_z \end{vmatrix}$$

$$J_3 = \begin{vmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{vmatrix}$$

Principal Strains and Directions

- The cubic equation $\epsilon^3 - J_1\epsilon^2 + J_2\epsilon - J_3 = 0$ has 3 real roots
- Each of this roots can be substituted to

$$(\epsilon_x - \epsilon)n_x + \epsilon_{yx}n_y + \epsilon_{zx}n_z = 0$$

$$\epsilon_{xy}n_x + (\epsilon_y - \epsilon)n_y + \epsilon_{zy}n_z = 0$$

$$\epsilon_{xz}n_x + \epsilon_{yz}n_y + (\epsilon_z - \epsilon)n_z = 0$$

  to get corresponding values ofn_x, n_y and n_z

- In order to avoid trivial solution, the condition ${n_x}^2 + {n_y}^2 + {n_z}^2 = 1$ is used with any two of the above equations to obtain n_x, n_y and n_z

Principal Strains and Directions - Excercise

The displacement field in micro units for a body is given by,

$$u = (x^2 + y)\mathbf{i} + (3 + z)\mathbf{j} + (x^2 + 2y)\mathbf{k}$$

Determine the principal strains at (3, 1, –2) and the direction of the minimum principal strain.

Solution :

The displacement components in micro units are, $u_x = (x^2 + y)$, $u_y = (3 + z)$ and $u_z = (x^2 + 2y)$

$$\epsilon_x = \frac{\partial u_x}{\partial x} = 2x, \; \epsilon_y = \frac{\partial u_y}{\partial y} = 0, \; \epsilon_z = \frac{\partial u_z}{\partial z} = 0$$

Principal Strains and Directions - Excercise

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}(0+1) = \frac{1}{2},$$
$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) = \frac{1}{2}(2+1) = \frac{3}{2},$$
$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}(0+2x) = x$$

At point (3, 1, –2) the strain components are therefore,

$$\epsilon_x = 6, \epsilon_y = 0, \epsilon_z = 0$$
$$\epsilon_{xy} = \frac{1}{2}, \epsilon_{yz} = \frac{3}{2}, \epsilon_{zx} = 3$$

Strain Invariants are, $J_1 = 6, J_2 = \frac{-23}{2}, J_3 = -9$

Principal Strains and Directions - Excercise

The Cubic equation is,

$$\epsilon^3 - 6\epsilon^2 - \frac{23}{2}\epsilon + 9 = 0$$

The roots of the above equation can be obtained as,
$\epsilon_1 = +7.39, \epsilon_2 = -2, \epsilon_3 = +0.61$
As a check, Invariants can be found out,
$J_1 = 6, J_2 = -11.49, J_3 = -9$
The minimum principal strain is -2. For this, direction cosines are
$n_x = 0.267, n_y = 0.534, n_z = -0.801$[1]

[1] Please refer the earlier exercise on principal stresses or textbook for details

Compatibility Conditions

- The displacement of a point can be represented by u_x, u_y, and u_z along the three axes x, y and z respectively.
- The deformation at a point is specified by the six strain components ϵ_x, ϵ_y, ϵ_z, ϵ_{xy}, ϵ_{yz} and ϵ_{zx}
- Determination of strains from displacements involves only differentiation
- Determination of three displacement components from six strain components needs **Compatibility Conditions** to be satisfied by strain components

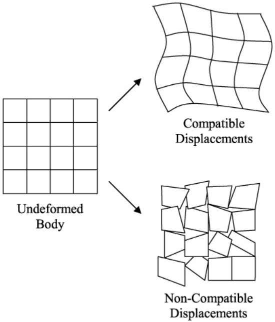

Figure: Deformations

Compatibility Conditions

These equations describe the relations between the components of strain and was put forward by **St. Venant**.
Consider the following relationships

$$\epsilon_x = \frac{\partial u_x}{\partial x} \quad \epsilon_y = \frac{\partial u_y}{\partial y} \quad \epsilon_z = \frac{\partial u_z}{\partial z}$$

Differentiating the first relation twice with respect to y and second with respect to x, we get

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u_x}{\partial y^2 \partial x} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right) \qquad \text{(i)}$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right) \qquad \text{(ii)}$$

Compatibility Conditions

Adding (i) and (ii), $\dfrac{\partial^2 \epsilon_x}{\partial y^2} + \dfrac{\partial^2 \epsilon_y}{\partial x^2} = \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right)$

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Similarly, considering ϵ_y, ϵ_z & γ_{yz} and ϵ_z, ϵ_x & γ_{zx} two more equations can be formed.

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

Compatibility Conditions

Considering the following relationships,

$$\gamma_{xy} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \quad \gamma_{yz} = \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \quad \gamma_{zx} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

Differentiating the first one with respect to z, second one with respect to x and last one with respect to y, we get

$$\frac{\partial \gamma_{xy}}{\partial z} = \left(\frac{\partial^2 u_y}{\partial z \partial x} + \frac{\partial^2 u_x}{\partial z \partial y}\right) \qquad \frac{\partial \gamma_{yz}}{\partial x} = \left(\frac{\partial^2 u_z}{\partial x \partial y} + \frac{\partial u_y}{\partial x \partial z}\right)$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \left(\frac{\partial^2 u_x}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial y \partial x}\right)$$

Compatibility Conditions

Adding the last two equations and subtracting the first equation we get,

$$\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = 2\frac{\partial^2 u_z}{\partial x \partial y}$$

Differentiating with respect to z,this becomes

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^3 u_z}{\partial x \partial y \partial z} = 2\frac{\partial^2 \epsilon_z}{\partial x \partial y}$$

Other equations can be obtained by cyclic change of subscripts.

Compatibility Conditions

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \epsilon_z}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \epsilon_x}{\partial y \partial z}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \epsilon_y}{\partial z \partial x}$$

These equations are called **Saint Venant's** equations of compatibility.

Compatibility Conditions-Excercises

Find the conditions needed to hold between the constants a_1 to a_6 so that the following is a possible strain field. $\epsilon_x = a_1(x^2 + y^2)$; $\epsilon_y = a_2(y^2 + z^2)$; $\epsilon_z = a_3(z^2 + x^2)$; $\gamma_{xy} = a_4 xy$; $\gamma_{yz} = a_5 yz$; $\gamma_{zx} = a_6 zx$

Compatibility Conditions-Excercises

Find the conditions needed to hold between the constants a_1 to a_6 so that the following is a possible strain field. $\epsilon_x = a_1(x^2 + y^2)$; $\epsilon_y = a_2(y^2 + z^2)$; $\epsilon_z = a_3(z^2 + x^2)$; $\gamma_{xy} = a_4 xy$; $\gamma_{yz} = a_5 yz$; $\gamma_{zx} = a_6 zx$

Solution: From compatibility conditions,

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$2a_1 + 0 = a_4$$

$$2a_1 = a_4 \tag{6.1}$$

Similarly,

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial x \partial y}$$

$$2a_2 + 0 = a_5$$

$$2a_2 = a_5 \tag{6.2}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial x \partial y}$$

$$2a_3 + 0 = a_6$$

$$2a_3 = a_6 \tag{6.3}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)=2\frac{\partial^2\epsilon_z}{\partial x\partial y}$$

$$\frac{\partial}{\partial z}\left(0+0+0\right)=0$$

This condition is identically satisfied.

$$\frac{\partial}{\partial x}\left(\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}-\frac{\partial\gamma_{yz}}{\partial x}\right)=2\frac{\partial^2\epsilon_x}{\partial x\partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}\right)=2\frac{\partial^2\epsilon_y}{\partial x\partial y}$$

These two conditions are also identically satisfied.
Hence, The conditions needed to be satisfied between constants a_1 to a_6 are,

$$2a_1=a_4,\quad 2a_2=a_5\quad and\quad 2a_3=a_6$$