Stress-Strain Relationships

Advanced Mechanics of Solids ME202

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Outline

Constitutive equations

- The relationships between the stress and strain components are termed **constitutive equations**.
- These relations depend on the manner in which the material resists deformation.
- The constitutive equations are mathematical descriptions of the physical phenomena based on experimental observations and established principles.

Generalized Hook's Law

Each of the six independent components of stress (**for a homogeneous linearly elastic material**) may be expressed as a linear function of the six components of strain and vice versa.

Six Stress–Strain equations:

$$\sigma_x = a_{11}\epsilon_{xx} + a_{12}\epsilon_{yy} + a_{13}\epsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{yz} + a_{16}\gamma_{zx}$$

$$\sigma_y = a_{21}\epsilon_{xx} + a_{22}\epsilon_{yy} + a_{23}\epsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{yz} + a_{26}\gamma_{zx}$$

$$\sigma_z = a_{31}\epsilon_{xx} + a_{32}\epsilon_{yy} + a_{33}\epsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{yz} + a_{36}\gamma_{zx}$$

$$\tau_{xy} = a_{41}\epsilon_{xx} + a_{42}\epsilon_{yy} + a_{43}\epsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{yz} + a_{46}\gamma_{zx}$$

$$\tau_{yz} = a_{51}\epsilon_{xx} + a_{52}\epsilon_{yy} + a_{53}\epsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{yz} + a_{56}\gamma_{zx}$$

$$\tau_{zx} = a_{61}\epsilon_{xx} + a_{62}\epsilon_{yy} + a_{63}\epsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{yz} + a_{66}\gamma_{zx}$$

Generalized Hook's Law

In matrix form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

Generalized Hook's Law

Similarly Six Strain-Stress equations can also be formed:

$$\epsilon_x = b_{11}\sigma_x + b_{12}\sigma_y + b_{13}\sigma_z + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{zx}$$

$$\epsilon_y = b_{21}\sigma_x + b_{22}\sigma_y + b_{23}\sigma_z + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{zx}$$

$$\epsilon_z = b_{31}\sigma_x + b_{32}\sigma_y + b_{33}\sigma_z + b_{34}\tau_{xy} + b_{35}\tau_{yz} + b_{36}\tau_{zx}$$

$$\gamma_{xy} = b_{41}\sigma_x + b_{42}\sigma_y + b_{43}\sigma_z + b_{44}\tau_{xy} + b_{45}\tau_{yz} + b_{46}\tau_{zx}$$

$$\gamma_{yz} = b_{51}\sigma_x + b_{52}\sigma_y + b_{53}\sigma_z + b_{54}\tau_{xy} + b_{55}\tau_{yz} + b_{56}\tau_{zx}$$

$$\gamma_{zx} = b_{61}\sigma_x + b_{62}\sigma_y + b_{63}\sigma_z + b_{64}\tau_{xy} + b_{65}\tau_{yz} + b_{66}\tau_{zx}$$

Generalized Hook's Law

In matrix form:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}$$

Equations for Linear Elastic Isotropic Solids

- Isotropic material means a material having identical values of a property in all directions.
- For isotropic materials it can be shown that only two independent elastic constants are involved in the generalised Hooke's law.

Lame's coefficients

For isotropic materials, normal stress along a particular direction is influenced by the normal strain along that direction as well as the volumetric strain,

$$\sigma_x = 2\mu\epsilon_x + \lambda\Delta$$

Where, Volumetric Strain $\Delta = \epsilon_x + \epsilon_y + \epsilon_z$

$$\sigma_x = 2\mu\epsilon_x + \lambda\left(\epsilon_x + \epsilon_y + \epsilon_z\right)$$

$$\sigma_y = 2\mu\epsilon_y + \lambda\left(\epsilon_x + \epsilon_y + \epsilon_z\right)$$

$$\sigma_z = 2\mu\epsilon_z + \lambda\left(\epsilon_x + \epsilon_y + \epsilon_z\right)$$

$$\tau_{xy} = \mu\gamma_{xy}, \quad \tau_{yz} = \mu\gamma_{yz}, \quad \tau_{zx} = \mu\gamma_{zx}$$

Where, μ and λ are called Lame's coefficients.

Relation between elastic constants

$$\tau_{xy} = \mu\gamma_{xy}, \quad \tau_{yz} = \mu\gamma_{yz}, \quad \tau_{zx} = \mu\gamma_{zx}$$

From Strength of materials,

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}$$

Where, G is the Modulus of Rigidity of the material.
Hence,

$$\boxed{\mu = G}$$

Relation between elastic constants

$$\epsilon_x = \frac{1}{2\mu}\left(\sigma_x - \lambda[\epsilon_x + \epsilon_y + \epsilon_z]\right) \tag{4.1}$$

$$[\epsilon_x + \epsilon_y + \epsilon_z] = \frac{[\sigma_x + \sigma_y + \sigma_z]}{(2\mu + 3\lambda)} \tag{4.2}$$

Putting (4.2) in (4.1)

$$\epsilon_x = \frac{1}{2\mu}\left(\sigma_x - \lambda\left[\frac{(\sigma_x + \sigma_y + \sigma_z)}{(2\mu + 3\lambda)}\right]\right)$$

Relation between elastic constants

$$\epsilon_x = \left[\frac{1}{2\mu}\left(1 - \frac{\lambda}{2\mu + 3\lambda}\right)\right]\sigma_x - \left(\frac{\lambda}{2\mu(2\mu + 3\lambda)}\right)\sigma_y - \left(\frac{\lambda}{2\mu(2\mu + 3\lambda)}\right)\sigma_z$$

This equation is of the form

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

Equating the coefficients of σ_x, we get

$$\boxed{E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}}$$

Relation between elastic constants

$$\epsilon_x = \left[\frac{1}{2\mu}\left(1 - \frac{\lambda}{2\mu + 3\lambda}\right)\right]\sigma_x - \left(\frac{\lambda}{2\mu(2\mu + 3\lambda)}\right)\sigma_y - \left(\frac{\lambda}{2\mu(2\mu + 3\lambda)}\right)\sigma_z$$

This equation is of the form

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

Equating the coefficients of σ_y, we get

$$\boxed{\nu = \frac{\lambda}{2(\mu + \lambda)}}$$

Relation among μ, λ, E and ν

$$\mu(=G) = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$$

Relation among μ, λ,E,ν and K

For hydrostatic state of stress, we have $\sigma_x = \sigma_y = \sigma_z = -p$. Applying this in equation (4.2) we get

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{(-p - p - p)}{(2\mu + 3\lambda)} \implies \frac{-p}{\epsilon_x + \epsilon_y + \epsilon_z} = \frac{(2\mu + 3\lambda)}{3}$$

since $(\epsilon_x + \epsilon_y + \epsilon_z) = \Delta$ is the volumetric strain, The ratio of pressure $(-p)$ to Volumetric strain is called Bulk modulus (K).

$$K = \frac{(2\mu + 3\lambda)}{3}$$

$$\boxed{K = \frac{E}{3(1 - 2\nu)}}$$

Relation among μ, λ,E,ν,G and K-Summary

$$\mu = G$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)}$$

$$\frac{9}{E} = \frac{3}{G} + \frac{1}{K}$$

Equations for Linear Elastic Isotropic Solids-Matrix form

For isotropic materials,

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}$$

Where E is Young's modulus, ν is Poisson's Ratio and G is modulus of Rigidity and $G = \dfrac{E}{2(1+\nu)}$

Equations for Linear Elastic Isotropic Solids-Matrix form

For isotropic materials,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

Where $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

Equations of Equilibrium

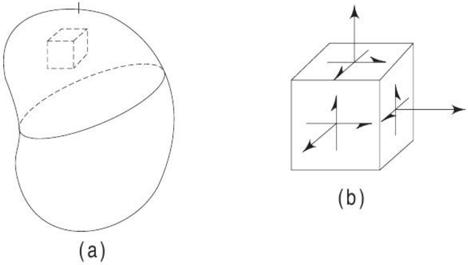

Figure: Isolated cubical element

Equations of Equilibrium

Three differential equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \gamma_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \gamma_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \gamma_z = 0$$

Let F_x, F_y, F_z be components of externally applied traction (force per unit area) at a boundary point P. Then internal resisting stress vector should match with the externally applied traction as the boundary condition.
Thus the set of boundary conditions for elasticity problem for externally applied traction are

$$\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z = F_x$$

$$\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z = F_y$$

$$\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = F_z$$

St.Venant's Principle for End Effects

Saint Venant's Principle of end loads:- The principle states that a change of loading distribution by a statically equivalent system of forces having same resultant force and couple on a small part of the surface of the body would give rise to localised changes in stress and strain only. Sufficiently away from this area the stress and strain fields will not be affected.

St.Venant's Principle for End Effects

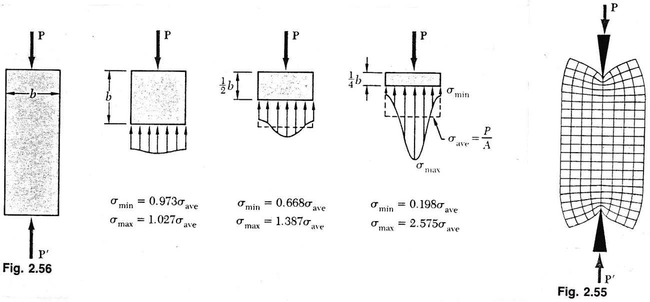

Figure: St.Venant's Principle

Uniqueness Theorem

Uniqueness Theorem:-Any physically realistic elasticity problem, defined by a set of governing equations and a set of boundary conditions will have one and only one solution

Uniqueness Theorem-Proof

To prove the theorem, we assume that there are two stress tensor fields σ_{ij} and σ'_{ij} satisfying the equilibrium equations and boundary conditions. Therefore

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \gamma_x = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \gamma_y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \gamma_z = 0
\end{aligned} \tag{7.1}$$

Uniqueness Theorem-Proof

$$
\begin{aligned}
\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z &= F_x \\
\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z &= F_y \\
\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z &= F_z
\end{aligned} \tag{7.2}
$$

Uniqueness Theorem-Proof

For the second set of solution field, we write

$$\begin{aligned}\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + \gamma_x &= 0\\ \frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau'_{yz}}{\partial z} + \gamma_y &= 0\\ \frac{\partial \tau'_{xz}}{\partial x} + \frac{\partial \tau'_{yz}}{\partial y} + \frac{\partial \sigma'_z}{\partial z} + \gamma_z &= 0\end{aligned} \tag{7.3}$$

$$\begin{aligned}\sigma'_x n_x + \tau'_{yx} n_y + \tau'_{zx} n_z &= F_x\\ \tau'_{xy} n_x + \sigma'_y n_y + \tau'_{zy} n_z &= F_y\\ \tau'_{xz} n_x + \tau'_{yz} n_y + \sigma'_z n_z &= F_z\end{aligned} \tag{7.4}$$

Uniqueness Theorem-Proof

Subtracting equation (7.3) from (7.1),

$$\begin{aligned}
\frac{\partial(\sigma_x - \sigma_x')}{\partial x} + \frac{\partial(\tau_{xy} - \tau_{xy}')}{\partial y} + \frac{\partial(\tau_{xz} - \tau_{xz}')}{\partial z} &= 0 \\
\frac{\partial(\tau_{xy} - \tau_{xy}')}{\partial x} + \frac{\partial(\sigma_y - \sigma_y')}{\partial y} + \frac{\partial(\tau_{yz} - \tau_{yz}')}{\partial z} &= 0 \qquad (7.5) \\
\frac{\partial(\tau_{xz} - \tau_{xz}')}{\partial x} + \frac{\partial(\tau_{yz} - \tau_{yz}')}{\partial y} + \frac{\partial(\sigma_z - \sigma_z')}{\partial z} &= 0
\end{aligned}$$

The above set of differential equations show that the difference of two stress fields satisfy the equilibrium equations with zero body force.

Subtracting equation (7.4) from (7.2),

$$
\begin{aligned}
(\sigma_x - \sigma'_x)n_x + (\tau_{xy} - \tau'_{xy})n_y + (\tau_{zx} - \tau'_{zx})n_z &= 0 \\
(\tau_{xy} - \tau'_{xy})n_x + (\sigma_y - \sigma'_y)n_y + (\tau_{yz} - \tau'_{yz})n_z &= 0 \\
(\tau_{zx} - \tau'_{zx})n_x + (\tau_{yz} - \tau'_{yz})n_y + (\sigma_z - \sigma'_z)n_z &= 0
\end{aligned}
\quad (7.6)
$$

The difference of two stress fields satisfy the condition of zero external traction. That difference of the two stress fields is such that there are no external loads. From the consideration of zero strain energy for zero external loads, the assumption of two stress fields satisfying the set of equations is not correct and there is only one stress field as solution.