Two Dimensional Problems in Elasticity

Advanced Mechanics of Solids ME202

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February 8, 2018

Outline

2-D problems in elasticity

- The 3X3 matrices of stress and strain at a point in 3D problems is simplified to 2X2 in 2D problems
- These problems are defined in a region over a plane.
- 2D problems in elasticity can be classified in to
  - Plane stress
  - Plane Strain
  - Axi-symmetric
- The stress and strain tensors of 2D problems in x-y plane are
$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \quad \text{and} \quad \epsilon = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{xy}}{2} & \epsilon_y \end{bmatrix}$$
Rest of the terms in 3D Tensor are either zero or related to those present in the 2X2 matrices by Hook's law

Plane Stress

- **Plane stress** is defined to be a state of stress in which normal stress σ_z, and shear stresses τ_{xz} and τ_{yz} directed perpendicular to x-y plane are assumed to be zero.
- Plane stress typically occurs in thin flat plates that are acted upon only by load forces that are parallel to them.

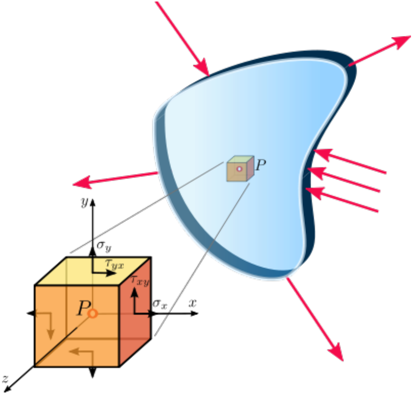

Figure: Plane stress

Plane Stress

- **Conditions for plane stress**
  $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, All other stress/strain components are independent of z-coordinate.
- **Generalized Hook's law**

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\epsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right) \quad \text{In plane stress} \quad \epsilon_z \neq 0$$

$$\epsilon_{xy} = \frac{1}{2\mu}\tau_{xy} \quad \text{Or} \quad \tau_{xy} = G\gamma_{xy}$$

Plane Stress

- **Equilibrium conditions**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0, \quad \text{Since} \quad \frac{\partial}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0, \quad \text{Since} \quad \frac{\partial}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0, \quad \text{Since} \quad \tau_{xz} = \tau_{yz} = b_z = 0, \frac{\partial}{\partial z} = 0$$

Plane Stress

- **Equilibrium conditions**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

- **Compatibility conditions**

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Plane Strain

- **Plane strain** is defined to be a state of strain in which the strain normal to the x-y plane,ϵ_z and shear strains γ_{xz} and γ_{yz} are assumed to be zero.
- This is possible, when the dimension of the solid is very large in z-direction compared to x and y directions, or when the displacement in a particular direction is arrested.

In plane strain, all the cross-sections parallel to the plane have the same stress pattern

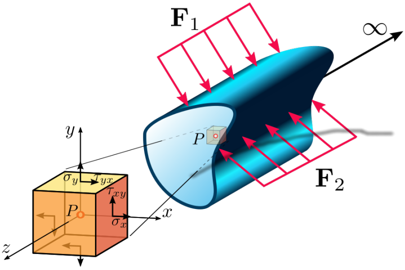

Figure: Plane strain

Plane Strain

- **Conditions for plane strain**
  $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0, u_z = 0$ All other stress/strain components are independent of z-coordinate.
- **Generalized Hook's law**

$$\epsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right) = 0 \implies \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right) = \frac{(1-\nu^2)}{E}\sigma_x - \frac{\nu(1+\nu)}{E}\sigma_y$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x - \nu\sigma_z\right) = -\frac{\nu(1+\nu)}{E}\sigma_x + \frac{(1-\nu^2)}{E}\sigma_y$$

$$\epsilon_{xy} = \frac{1}{2\mu}\tau_{xy} \quad \text{Or} \quad \tau_{xy} = G\gamma_{xy}$$

Plane Strain

- **Equilibrium conditions**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

- **Compatibility conditions**

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Equivalance between Plane stress and Plane strain

- They both seek solutions for σ_x, σ_y, τ_{xy} and ϵ_x, ϵ_y, ϵ_{xy} as a function of x and y.
- They satisfy the same equilibrium and compatibility conditions.
- Only difference is in Generalized Hook's law.

Plane Stress

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\epsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right), \quad \epsilon_{xy} = \frac{1}{2\mu}\tau_{xy}$$

Plane Strain

$$\epsilon_x = \frac{(1-\nu^2)}{E}\sigma_x - \frac{\nu(1+\nu)}{E}\sigma_y$$

$$\epsilon_y = -\frac{\nu(1+\nu)}{E}\sigma_x + \frac{(1-\nu^2)}{E}\sigma_y$$

$$\epsilon_z = 0, \quad \epsilon_{xy} = \frac{1}{2\mu}\tau_{xy}$$

Stress compatibility equation

Five out of the six compatibility equations are exactly satisfied by the components of strain of 2D problems. The only one compatibility equation that requires agreement is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Using $\gamma_{xy} = \frac{1}{G}\tau_{xy}$ and $\frac{1}{G} = \frac{2(1+\nu)}{E}$, we get

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{3.1}$$

Stress compatibility equation

We have the equilibrium equations in 2D

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \tag{3.2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \tag{3.3}$$

Differentiating (3.2) with respect to x, and (3.3) with respect to y and rearranging,

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial b_x}{\partial x} \tag{3.4}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial b_y}{\partial y} \tag{3.5}$$

Stress compatibility equation

Adding (3.4) and (3.5) $\implies$

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \tag{3.6}$$

By putting (3.6) in (3.1) $\implies$

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = -\frac{(1+\nu)}{E}\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \tag{3.7}$$

Stress compatibility equation

Rearranging (3.7), we get

$$E\left(\frac{\partial^2 \epsilon_x}{\partial y^2}+\frac{\partial^2 \epsilon_y}{\partial x^2}\right)+(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2}+\frac{\partial^2 \sigma_y}{\partial y^2}\right) = -(1+\nu)\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right) \quad (3.8)$$

Now, Hook's law can be applied to relate ϵ_x and ϵ_y in terms of σ_x and σ_y and final equation will be different for plane stress and plane strain problems as Hook's law expressions are different.

Stress compatibility equation-Plane Stress

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \implies E\epsilon_x = (\sigma_x - \nu\sigma_y) \tag{3.9}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \implies E\epsilon_y = (\sigma_y - \nu\sigma_x) \tag{3.10}$$

Differentiating (3.9) partially w.r.t x and (3.10) partially w.r.t y,

$$E\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} \tag{3.11}$$

$$E\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} \tag{3.12}$$

Stress compatibility equation-Plane Stress

Adding (3.11) and (3.12) and rearranging,

$$E\left(\frac{\partial^2\epsilon_x}{\partial y^2}+\frac{\partial^2\epsilon_y}{\partial x^2}\right) = \left(\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}\right)-\nu\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}\right) \quad (3.13)$$

Substituting (3.13) in (3.8),

$$\left(\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}\right)-\nu\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}\right)+(1+\nu)\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}\right) = -(1+\nu)\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right) \quad (3.14)$$

Stress compatibility equation-Plane Stress

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial y^2} = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.15)$$

$$\frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y) + \frac{\partial^2}{\partial y^2}(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.16)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.17)$$

Stress compatibility equation-Plane Stress

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \tag{3.18}$$

Where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the Laplacian operator.
Now, equations (3.2), (3.3) and the equation (3.18) make three equations for solving three unknowns σ_x, σ_y and τ_{xy} of a plane stress elasticity problem. Equation (3.18) is called ***Stress compatibility equation***

Stress compatibility equation-Plane Strain

$$\epsilon_x = \frac{(1-\nu^2)}{E}\sigma_x - \frac{\nu(1+\nu)}{E}\sigma_y \implies$$

$$E\epsilon_x = (1+\nu)[(1-\nu)\sigma_x - \nu\sigma_y] \quad (3.19)$$

$$\epsilon_y = -\frac{\nu(1+\nu)}{E}\sigma_x + \frac{(1-\nu^2)}{E}\sigma_y \implies$$

$$E\epsilon_y = (1+\nu)[-\nu\sigma_x + (1-\nu)\sigma_y] \quad (3.20)$$

Differentiating (3.19) w.r.t y and (3.20) w.r.t x we get,

$$E\frac{\partial^2\epsilon_x}{\partial y^2} = (1+\nu)\left[(1-\nu)\frac{\partial^2\sigma_x}{\partial y^2} - \nu\frac{\partial^2\sigma_y}{\partial y^2}\right] \quad (3.21)$$

$$E\frac{\partial^2\epsilon_y}{\ldots} \ldots \left[\ldots \frac{\partial^2\sigma_y}{\ldots} \ldots \frac{\partial^2\sigma_x}{\ldots} \right] \quad (3.22)$$

Stress compatibility equation-Plane Strain

(3.21)+(3.22) $\Longrightarrow$

$$E\left(\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}\right) =$$

$$(1+\nu)\left[(1-\nu)\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right] \quad (3.23)$$

Substituting (3.23) in (3.8) we get,

$$(1+\nu)\left[(1-\nu)\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right] +$$

$$(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.24)$$

Stress compatibility equation-Plane Strain

$$\left[(1-\nu)\frac{\partial^2\sigma_x}{\partial y^2} - \nu\frac{\partial^2\sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2\sigma_y}{\partial x^2} - \nu\frac{\partial^2\sigma_x}{\partial x^2}\right] + \left(\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2}\right) = -\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.25)$$

$$(1-\nu)\frac{\partial^2\sigma_x}{\partial y^2} + (1-\nu)\frac{\partial^2\sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2\sigma_y}{\partial x^2} + (1-\nu)\frac{\partial^2\sigma_x}{\partial x^2} = -\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.26)$$

$$\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial x^2} = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.27)$$

Stress compatibility equation-Plane Strain

$$\frac{\partial^2}{\partial x^2}\left(\sigma_x + \sigma_y\right) + \frac{\partial^2}{\partial y^2}\left(\sigma_x + \sigma_y\right) = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.28)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.29)$$

$$\nabla^2\left(\sigma_x + \sigma_y\right) = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) \quad (3.30)$$

This is Stress compatibility equation of plane strain problem.

Stress compatibility equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

Plane Stress

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

Plane Strain

$$\nabla^2 (\sigma_x + \sigma_y) = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

Airy's stress function-Zero body forces

- Introduce a scalar function $\phi(x, y)$, and write different components of stress tensor as derivatives of ϕ
- Assume zero body force, $b_x = b_y = 0$ then the equilibrium conditions becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

- These can be automatically satisfied by defining

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{4.1}$$

The equilibrium conditions can be easily verified.

Airy's stress function-Zero body forces

- So only PDF we need to worry about is the compatibility condition.

  Plane Stress:

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

  plane Strain

$$\nabla^2 (\sigma_x + \sigma_y) = \frac{-1}{(1-\nu)}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

- In case of Zero body forces both the above equations reduces to,

$$\nabla^2 (\sigma_x + \sigma_y) = 0 \tag{4.2}$$

Airy's stress function-Zero body forces

- Substituting (4.1) in (4.2), we get

$$\nabla^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0$$

$$\nabla^2 \nabla^2 \phi = 0 \quad Or \quad \nabla^4 \phi = 0$$

Where, $\nabla^4 = \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4} \right)$

Airy's stress function-in presence of body forces

- When the body force can be expressed as the derivatives of a potential

$$b_x = \frac{\partial V}{\partial x}, \quad b_y = \frac{\partial V}{\partial y} \tag{4.3}$$

- Then Airy's stress function can be defined as

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - V, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - V, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{4.4}$$

The equilibrium conditions can be easily verified.

Airy's stress function-in presence of body forces

- The equilibrium equations can be easily verified.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

- The compatibility conditions become,

Plane stress

$$\nabla^4 \phi = (1 - \nu)\nabla^2 V$$

Plane stain

$$\nabla^4 \phi = \left(\frac{1 - 2\nu}{1 - \nu}\right) \nabla^2 V$$

Polynomial of Degree One - An unstressed body

Consider $\phi(x, y) = a_1 x + b_1 y$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

The components of stress are zero. It represents an unstressed body.

Polynomial of Degree Two

Consider $\phi(x, y) = a_2x^2 + b_2xy + c_2y^2$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 2c_2, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2a_2, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0$$

The components of stress are constant over the region(except shear). It represents a constant stress field.

If $a_2 = b_2 = 0$ and $c_2 = \frac{F}{2A}$, $\phi(x, y) = \frac{Fy^2}{2A}$,

$\sigma_x = \frac{F}{A}, \quad \sigma_y = 0, \quad \tau_{xy} = 0$

This represents stress in an axial force member of area A applied by an axial force F

Polynomial of Degree Three

Consider $\phi(x, y) = a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3$

$$\sigma_x = 2c_3x + 6d_3y, \quad \sigma_y = 6a_3x + 2b_3y, \quad \tau_{xy} = -2(b_3x + c_3y)$$

The components of stresses represents a linearly varying stress field.
If $a_3 = b_3 = c_3 = 0$ and $d_3 = \dfrac{M}{6I}$, $\phi(x, y) = \dfrac{My^3}{6I}$,
$\sigma_x = \dfrac{My}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = 0$
This represents stresses in simple (pure) bending of a beam applied by a bending moment M.

Polynomial of Degree Four

Consider $\phi(x,y) = a_4x^4 + b_4x^3y + c_4x^2y^2 + d_4xy^3 + e_4y^4$
For this function to represent a stress function, it should satisfy

$$\nabla^4\phi = \left(\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4}{\partial\phi x^2y^2} + \frac{\partial^4\phi}{\partial y^4}\right) = 0$$

Substituting the partial derivatives,

$$24a_4 + 8c_4 + 24e_4 = 0 \therefore 3a_4 + 3e_4 + c_4 = 0$$

Solution for bending of a cantilever with an end load

$$\phi(x, y) = axy + bxy^3$$

Clearly this function satisfies biharmonic equation of stress compatibility

$$\frac{\partial \phi}{\partial y} = ax + 3bxy^2$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6bxy$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -(a + 3by^2)$$

Solution for bending of a cantilever with an end load

Boundary conditions of the problem are:

1. Stress free top and bottom layers,
   i.e. $\tau_{xy}(y = \pm h)$
   Using this, we get $-(a + 3bh^2) = 0 \implies b = \frac{-a}{3h^2}$

$$\tau_{xy} = -a\left(1 - \frac{y^2}{h^2}\right)$$

2. Sum of total shear force on any cross section is equal to applied load P, i.e. $\int_h^{-h} \tau_{xy} t dy = P$

Solution for bending of a cantilever with an end load

$$\int_{-h}^{h} \tau_{xy} t dy = P$$

$$-\int_{-h}^{h} a\left(1-\frac{y^2}{h^2}\right) t dy = -at\int_{-h}^{h}\left(1-\frac{y^2}{h^2}\right) dy$$

$$= -at\left(y-\frac{y^3}{3h^2}\right)_{-h}^{h}$$

$$= -2at\left(h-\frac{h}{3}\right) = -\frac{4ath}{3} = P$$

$$\implies a = -\frac{3P}{4th}, \quad b = \frac{P}{4th^3}$$

Solution for bending of a cantilever with an end load

$$\phi(x,y) = \frac{P}{4th}\left[\frac{xy^3}{h^2} - 3xy\right]$$

$$\sigma_x = \frac{3P}{2th^3}xy$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{3P}{4th^3}(h^2 - y^2)$$